February 7, 2008

Noncommutative Field Theory on Yang's Space-Time Algebra, Covariant Moyal Star Product and Matrix Model

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ABSTRACT

Noncommutative field theory on Yang's quantized space-time algebra (YSTA) is studied. It gives a theoretical framework to reformulate the matrix model as quantum mechanics of D_0 branes in a Lorentz-covariant form. The so-called kinetic term ($\sim \hat{P_i}^2$) and potential term ($\sim [\hat{X}_i, \hat{X}_j]^2$) of D_0 branes in the matrix model are described now in terms of Casimir operator of $SO(D,1)$, a subalgebra of the primary algebra $SO(D+1,1)$ which underlies YSTA with two contraction-parameters, λ and R. D-dimensional noncommutative space-time and momentum operators $\hat{X}_\mu$ and $\hat{P}_\mu$ in YSTA show a distinctive spectral structure, that is, space-components $\hat{X}_i$ and $\hat{P}_i$ have discrete eigenvalues, and time-components $\hat{X}_0$ and $\hat{P}_0$ continuous eigenvalues, consistently with Lorentz-covariance. According to the method of Lorentz-covariant Moyal star product proper to YSTA, the field equation of D_0 brane on YSTA is derived in a nontrivial form beyond simple Klein-Gordon equation, which reflects the noncommutative space-time structure of YSTA.

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1. Introduction

Along with the development of the matrix model of M-theory,[1] a lot of field theories on noncommutative space-time variables have been proposed.[2] In the most of them, the noncommutativity of space-time variables $\hat{x_i}$ are simply assumed in the form

$$[\hat{x_i}, \hat{x_j}] = i\theta_{ij}, \tag{1.1}$$

with an anti-symmetric constant tensor θ_{ij}. Field theories on noncommutative space-time are there related to a local field theory on commutative space-time by making use of the method of Moyal star product. Under this procedure, the propagator of field remains in most cases to be that of the commutative case and the noncommutativity of space-time is solely reflected on the appearance of nonlocal form-factor coming from Moyal star operation, $\exp\,(i/2\, \frac{\overleftarrow{\partial}}{\partial x_i}\theta_{ij}\frac{\overrightarrow{\partial}}{\partial x_j})$, in interaction terms among local fields.

At this point, we wonder why the propagator by itself does not reflect the noncommutative structure of space-time expressed by (1.1). One easily finds out that the situation simply comes from an assumption that the kinetic term of noncommutative field Φ is given by $\sim \frac{\partial \Phi}{\partial x_\mu} * \frac{\partial \Phi}{\partial x_\mu}$ under Moyal star product mentioned above. This problem will be our important concern in what follows.

In the present paper, we wish to extend our previous attempts,[3][4][5] to construct a noncommutative field theory of the matrix model, which is developed on a covariant quantized space-time algebra early proposed by C.N. Yang[6] immediately after H.S. Snyder,[7] in place of (1.1). As was emphasized in ref.4, Yang's quantized space-time algebra (YSTA), not simply as a toy model of noncommutative space-time algebra, but has several important characteristics.

First, as was pointed out in ref.5, it should be noticed that, YSTA has a common symmetry with D-dimensional (Euclidean) CFT, i.e., $SO(D+1,1)$ with two extra dimensions. It gives a possibility of regularizing a short distance behavior or divergences familiar in the so-called IR/UV connection in AdS/CFT or dS/CFT

correspondence, by virtue of discrete spectral structure of space-time and momentum operators of YSTA, which CFT lacks entirely.

Second, as will be shown in the present paper, one finds that YSTA is well fitted for covariant description of the matrix model. In fact, the so-called kinetic term ($\sim \hat{P}_i^{\,2}$) and potential term ($\sim [\hat{X}_i, \hat{X}_j]^2$) of D_0 branes in the matrix model, which are usually understood to originate in Yang-Mills gauge symmetry, are described now in terms of Casimir operator of $SO(D,1)$, a subalgebra of the primary algebra $SO(D+1,1)$ which underlies YSTA with two contraction-parameters, λ and R. According to the method of covariant Moyal star product proper to YSTA, we successfully derive D_0 brane field equation on YSTA in a nontrivial form beyond simple Klein-Gordon equation: It clearly reflects the quantized noncommutative space-time structure of YSTA, in contrast to most of familiar noncommutative field theories as mentioned in the beginning.

The present paper is organized as follows. In section 2, we shortly recapitulate the algebraic structure of Yang's quantized space-time algebra (YSTA) contracted from $SO(D+1,1)$ and examine characteristics of the so-called quasi-regular representation,[8] a kind of unitary infinite dimensional representation of SO(D+1,1). We investigate translation operation in YSTA in connection with momentum operators $\hat{P}_\mu$ and clarify how commutative space-time, Heisenberg's uncertainty relation, translations and so forth in the ordinary quantum mechanics are restored in YSTA. It will be found that they are all together restored in appropriate limiting values of contraction- parameters λ and R in a large limit of discrete eigenvalues of reciprocity operator of YSTA, which will be called in what follows, the "quantum-mechanical limit of YSTA".

In section 3, we try to rewrite the matrix model covariantly in terms of noncommutative field theory on YSTA, by making use of the result in section 2. In order to see a short distance behavior of the present noncommutative field theory, we derive the field equation of D_0 brane by means of covariant Moyal star product proper to YSTA. Preliminary considerations on the new equation are given.

The final section is devoted to discussions and concluding remarks.

2. YSTA and its Quasi-Regular Representation

Yang's quantized space-time algebra (YSTA)[6] was proposed to modify the original Snyder's quantized space-time algebra[7] to be translation-invariant, in addition to Lorentz-invariance which holds in both theories.

D-dimensional YSTA is contracted from $SO(D+1,1)$ algebra with generators $\hat{\Sigma}_{MN}$;

$$\hat{\Sigma}_{MN} \equiv i\ (q_M \frac{\partial}{\partial q_N} - q_N \frac{\partial}{\partial q_M}) \tag{2.1}$$

which work on $(D+2)$-dimensional parameter space q_M $(M = \mu, a, b)$ satisfying

$$-q_0^2 + q_1^2 + ... + q_{D-1}^2 + q_a^2 + q_b^2 = R^2. \tag{2.2}$$

Here, $q_0 = -iq_D$ and $M = a, b$ denote two extra dimensions with space-like metric signature.

D-dimensional space-time and momentum operators, $\hat{X}_\mu$ and $\hat{P}_\mu$, with $\mu = 1, 2, ..., D$, are defined by

$$\hat{X}_\mu \equiv \lambda\ \hat{\Sigma}_{\mu a} \tag{2.3}$$

$$\hat{P}_\mu \equiv \hbar/R\ \hat{\Sigma}_{\mu b}, \tag{2.4}$$

together with D-dimensional angular momentum operator $\hat{M}_{\mu\nu}$

$$\hat{M}_{\mu\nu} \equiv \hbar \hat{\Sigma}_{\mu\nu} \tag{2.5}$$

and the so-called reciprocity operator

$$\hat{N} \equiv \lambda/R\ \hat{\Sigma}_{ab}. \tag{2.6}$$

In the above expressions, the so-called contraction-parameters λ in (2.3) and R in (2.4) are to be fundamental constants of YSTA, as will be seen below.

Operators $(\hat{X}_\mu, \hat{P}_\mu, \hat{M}_{\mu\nu}, \hat{N})$ defined above constitute Yang's quantized space-time algebra, YSTA:

$$[\hat{X}_\mu, \hat{X}_\nu] = -i\lambda^2/\hbar \hat{M}_{\mu\nu} \tag{2.7}$$

$$[\hat{P}_\mu, \hat{P}_\nu] = -i\hbar/R^2 \ \hat{M}_{\mu\nu} \tag{2.8}$$

$$[\hat{X}_\mu, \hat{P}_\nu] = -i\hbar \hat{N} \delta_{\mu\nu} \tag{2.9}$$

$$[\hat{N}, \hat{X}_\mu] = -i\lambda^2/\hbar \hat{P}_\mu \tag{2.10}$$

$$[\hat{N}, \hat{P}_\mu] = -i\hbar/R^2 \ \hat{X}_\mu, \tag{2.11}$$

with familiar relations among $\hat{M}_{\mu\nu}$ $'s$ omitted.

At this point, it is important to notice the following simple fact that $\hat{\Sigma}_{MN}$ with M, N being the same metric signature have discrete eigenvalues and those with M, N being opposite metric signature have continuous eigenvalues, as was shown explicitly in ref. 4. For the subsequent arguments, let us reconfirm this fact through space-time operators $\hat{X}_\mu$,

$$\hat{X}_i = \lambda \hat{\Sigma}_{ia} = i\lambda(-q_i \frac{\partial}{\partial q_a} + q_a \frac{\partial}{\partial q_i}), \tag{2.12}$$

$$\hat{X}_0 = \lambda \hat{\Sigma}_{0a} = i\lambda(q_0 \frac{\partial}{\partial q_a} + q_a \frac{\partial}{\partial q_0}). \tag{2.13}$$

They are rewritten as

$$\begin{aligned} \hat{X}_i &= \lambda \frac{1}{i} \frac{\partial}{\partial \alpha_i}, \\ \hat{X}_0 &= \lambda \frac{1}{i} \frac{\partial}{\partial \alpha_0} \end{aligned} \tag{2.14}$$

with α_i $(0 \leq |\alpha_i| \leq \pi)$ and α_0 $(0 \leq |\alpha_0| \leq \infty)$ defined through $q_i/q_a = \tan\alpha_i$ and $q_0/q_a = \tanh\alpha_0$, respectively. As was remarked above, $\hat{X}_i$ and $\hat{X}_0$ in (2.14),

respectively, have discrete and continuous eigenvalues, λm_i with $\pm$ integer m_i and real number t. The corresponding eigenfunctions are give by

$$\begin{aligned}\left|\hat{\Sigma}_{ia} = m_i\right\rangle \sim \exp(i\ m_i \alpha_i) &= \exp[i\ m_i\ \tan^{-1}(q_i/q_a)] \\ &= [(q_a - iq_i)/(q_a + iq_i)]^{im_i/2}\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\left|\hat{\Sigma}_{0a} = t/\lambda\right\rangle \sim \exp[i\ (t/\lambda)\ \alpha_0] &= \exp[i\ (t/\lambda)\ \tanh^{-1}(q_0/q_a)] \\ &= [(q_a + q_0)/(q_a - q_0)]^{it/(2\lambda)},\end{aligned} \tag{2.16}$$

where normalization constants are omitted.

Needless to say, the eigenstate $\left|\hat{\Sigma}_{0a} = t/\lambda\right\rangle$ given by (2.16) is concerned with a boost operator $\hat{\Sigma}_{0a}$ of $S(D+1,1)$, and its continuous eigenvalue t $(/\lambda)$ is to be identified with eigenvalue t of time operator $\hat{X}_{0a}$, (2.13). This fact implies that Yang's space-time algebra (YSTA) presupposes for its representation space to take representation bases like $\left|\hat{\Sigma}_{0a} = t/\lambda, n...\right\rangle$, where $n...$ denotes eigenvalues of maximal commuting set of compact subalgebra of $SO(D+1,1)$ which are commutative with $\hat{\Sigma}_{0a}$, for instance, $\hat{\Sigma}_{b1}$, $\hat{\Sigma}_{23}$,..., $\hat{\Sigma}_{89}$, when $D = 11$.*)

Indeed, an infinite dimensional linear space expanded by $\left|\hat{\Sigma}_{0a} = t/\lambda, n...\right\rangle$ mentioned above provides a Hilbert space, called hereafter Hilbert space I according to ref.4, which becomes a representation space of unitary infinite dimensional representation of $SO(D+1,1)$ and of YSTA. It is a kind of "quasi-regular representation" [8] of SO(D+1,1), which is reducible to the familiar regular representations of $SO(D+1,1)$ algebra realized on its compact subalgebra. We expect the following expansion formula to hold

$$\left|\hat{\Sigma}_{0a} = t/\lambda, n...\right\rangle = \sum_{\sigma' s} \sum_{jm...} C^{\sigma,n...}_{jm...}(t/\lambda)\ \left|\ \sigma' s; j(j+1), m, ...\right\rangle, \tag{2.17}$$

where $|\sigma' s; j(j+1), m, ...\rangle$ on the right hand side describe familiar unitary irre-

*) It corresponds, in the case of unitary representation of Lorentz group $SO(3,1)$, to take K_3 ($\sim \Sigma_{03}$) and J_3 ($\sim \Sigma_{12}$) to be diagonal, which have continuous and discrete eigenvalues, respectively, instead of $\mathbf{J}^2$ and J_3 in a regular representation.

ducible representation bases of $SO(D+1,1)$ with Casimir invariants $\sigma' s$, continuous or discrete in general, over which the summation $\Sigma_{\sigma' s}$ ranges. Expansion coefficients $C^{\sigma' s}_{jm...}(t/\lambda)$ are to be calculated by making use of the functional form (2.16) for $\left|\hat{\Sigma}_{0a} = t/\lambda, ...\right\rangle$ on the left hand side.

Before closing this section, let us remark on the translation operation in YSTA, which was one of central motivation of YSTA beyond the original Snyder's quantized space-time algebra, where this operation did not work. Let us define D-dimensional translation operator $\hat{T}$ with infinitesimal parameters α_μ by

$$\hat{T}(\alpha_\mu) = \exp\ i\ (\alpha_\mu\ \hat{P}_\mu). \tag{2.18}$$

One finds that this operator induces infinitesimal transformation on $\hat{X}_\mu$

$$\hat{X}_\mu \to \hat{X}_\mu + \alpha_\mu\ \hat{N} \tag{2.19}$$

together with

$$\hat{N} \to \hat{N} - \alpha_\mu\ \hat{X}_\mu / R^2. \tag{2.20}$$

This result is well understood, if one notices that the momentum operator $\hat{P}_\mu$ is nothing but generator of infinitesimal rotation on $b-\mu$ plane, $\hat{\Sigma}_{\mu b}$ given in (2.4).

However, let us here notice that the reciprocity operator $\hat{N}(=\lambda R^{-1}\hat{\Sigma}_{ab})$ defined in (2.6) is an operator with discrete eigenvalues $n\ (\lambda R^{-1})$, n being $\pm$ integer and the displacement $\alpha_\mu \hat{N}$ in (2.19) is noncommutative with $\hat{X}_\mu$. Therefore, it is important to see in what limit ordinary translations familiar in quantum mechanics may be exactly restored. Indeed, one finds them in a following limit of contraction-parameters, λ and R,

$$\begin{aligned} &\lambda \to 0 \\ &R \to \infty, \end{aligned} \tag{2.21}$$

in conformity with a condition

$$\hat{N}(=\lambda R^{-1}\hat{\Sigma}_{ab}) \to 1. \tag{2.22}$$

In fact, the condition (2.22) under the limit (2.21) necessitates a large limit of discrete eigenvalues of $\hat{\Sigma}_{ab}$ in order for $\hat{N}$ to survive with nonvanishing value 1. Furthermore, one finds that the above limit completely restores the ordinary commutative space-time in addition to Heisenberg's uncertainty relation, as seen in (2.7) to (2.11). This fact reminds us Bohr's correspondence principle at the birth of quantum mechanics, that is, quantum mechanics tends to classical mechanics in a large limit of quantum numbers.

3. Noncommutative Field Theory on YSTA, Covariant Moyal Star Product and Matrix Model

Now we are in a position to rewrite the matrix model[1] in terms of noncommutative quantum field theory on YSTA. The so-called kinetic term and potential term of D_0 branes or D-particles are described in the following form

$$L = \alpha \ \mathrm{Tr}\ (P_i)^2 - \beta \ \mathrm{Tr}\ [X_i, X_j]^2, \tag{3.1}$$

where X_i and P_i are, respectively, $N \times N$ matrices of position and momentum of N D-particles, and coefficient constants α and β are given in terms of fundamental constants of string theory.

At this point, we wish to rewrite the above Lagrangian covariantly by taking into consideration the following relation

$$\hat{\Sigma}_{KL}^2 = 2\ \hat{\Sigma}_{\mu b}^2 + \hat{\Sigma}_{\mu\nu}^2 = 2(R/\hbar)^2 \hat{P}_\mu^2 - \lambda^{-4}\ [\hat{X}_\mu, \hat{X}_\nu]^2, \tag{3.2}$$

where $\hat{\Sigma}_{KL}^2$ denotes $(D+1)$-dimensional Casimir operator of $SO(D,1)$, with K, L ranging over μ and b.

Indeed, in place of L in (3.1), we propose an action $\bar{L}$ of noncommutative field theory on YSTA by

$$\begin{aligned}\bar{L} &= A \ \mathrm{Tr}\ \big[\ [\hat{\Sigma}_{KL}, \hat{D}^{\dagger}]\ [\hat{\Sigma}_{KL}, \hat{D}]\ \big] \\ &= A' \ \mathrm{Tr}\ \big[\ 2\ (R^2/\hbar^2)\ [\hat{P}_{\mu}, \hat{D}^{\dagger}]\ [\hat{P}_{\mu}, \hat{D}] - \lambda^{-4}\ [\ [\hat{X}_{\mu}, \hat{X}_{\nu}],\ \hat{D}^{\dagger}]\ [[\hat{X}_{\mu}, \hat{X}_{\nu}], \hat{D}]\ \big],\end{aligned} \tag{3.3}$$

where $\hat{D}$ and $\hat{D}^{\dagger}$, respectively, denote D_0 brane field operator on YSTA and its hermitian conjugate. These fields on YSTA must be also operators working on Hilbert space I discussed in section 2, that is, the representation space of $\hat{\Sigma}_{AB}$. In ref.4, we studied the time development of D-field operator $\hat{D}$ under the assumption that in light-cone frame, $\hat{D}$ becomes diagonal with respect to light-cone time, or $[X_+, \hat{D}] = 0$.*)

In the present paper, however, in order to see a short distance behavior of the present noncommutative field theory, we investigate the field equation of D_0 brane by making use of the familiar method of Moyal star product of noncommutative field theory stated in Introduction. At this point, it is important to note that in the present case, Moyal star product concerning Lorentz-covariant YSTA becomes also Lorentz-covariant.

It is well known that if a system is described by $2n$ canonical variables, q_A and p_A $(A = 1, 2, ..., n)$, or the corresponding quantized variables, $\hat{q}_A$ and $\hat{p}_A$, operator product of any two functions $\hat{F}(\hat{q}, \hat{p})$ and $\hat{G}(\hat{q}, \hat{p})$ is accompanied with Moyal star product of the corresponding classical functions, $F(q, p)$ and $G(q, p)$, that is,

$$\begin{aligned}\hat{F}(\hat{q}, \hat{p})\ \hat{G}(\hat{q}, \hat{p})\ &\sim F(q, p) * G(q, p) \\ &\equiv\ F(q, p)\ \exp\ \frac{i}{2}\ \Big(\ \frac{\overleftarrow{\partial}}{\partial q_A}\ \frac{\overrightarrow{\partial}}{\partial p_A} - \frac{\overleftarrow{\partial}}{\partial p_A}\ \frac{\overrightarrow{\partial}}{\partial q_A}\Big)\ G(q, p),\end{aligned} \tag{3.4}$$

when

$$\begin{aligned}\hat{F}(\hat{q}, \hat{p}) &\sim F(q, p), \\ \hat{G}(\hat{q}, \hat{p}) &\sim G(q, p).\end{aligned} \tag{3.5}$$

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*) In ref.4, the potential term was regarded as interaction term in a naive sense, rewriting it as quartic term of D_0 brane field, while the argument for (light-cone) time development of field used there is still applicable in the present case.

In our present case, it is important to note that basic quantities of YSTA, $\hat{\Sigma}_{MN}$ in (2.1), are expressed by

$$\hat{\Sigma}_{MN} = (-\hat{q}_M \hat{p}_N + \hat{q}_N \hat{p}_M), \tag{3.6}$$

so the relation (3.4) turns to give the following covariant Moyal star product for any two functions, $\hat{F}$ and $\hat{G}$ on $\hat{\Sigma}(\hat{q}, \hat{p})$,

$$\begin{aligned} \hat{F}(\hat{\Sigma})\ \hat{G}(\hat{\Sigma}) \sim & F(\Sigma) * G(\Sigma) \\ & = F(\Sigma)\ \exp\ \frac{i}{2}\ (\overleftarrow{\frac{\partial}{\partial \Sigma_{MN}}}\ \Sigma_{NO}\ \overrightarrow{\frac{\partial}{\partial \Sigma_{OM}}})\ G(\Sigma), \end{aligned} \tag{3.7}$$

where $\Sigma_{MN} = (-q_M p_N + q_N p_M)$.

Furthermore, by remarking that $[\hat{F}, \hat{G}]$ in (3.3) is replaced by Moyal bracket, $\{F, G\}_M \equiv F * G - G * F$, we find

$$\begin{aligned} [\hat{\Sigma}_{KL}, \hat{D}(\hat{\Sigma})] & \sim \{\Sigma_{KL}, D(\Sigma)\}_M \\ & = (-\Sigma_{aK} \frac{\partial}{\partial \Sigma_{aL}} + \Sigma_{aL} \frac{\partial}{\partial \Sigma_{aK}}) D(\Sigma). \end{aligned} \tag{3.8}$$

It should be noted that at the last step we assumed, self-consistently with D_0 brane field equation to be derived below, that D_0 brane field operator depends only on $\hat{\Sigma}_{aK}$, that is, $\hat{X}_\mu$ and $\hat{N}$, a minimum set which includes space-time variables and allows translation operations given by (2.19) and (2.20).

Applying variational principle on the classical action corresponding to $\bar{L}$, we obtain the field equation of D_0 brane

$$[\ {\Sigma_{aK}}^2\ (\frac{\partial}{\partial \Sigma_{aL}})^2 - (\Sigma_{aK} \frac{\partial}{\partial \Sigma_{aK}})^2 - (D-1) \Sigma_{aK} \frac{\partial}{\partial \Sigma_{aK}}\ ]\ D(\Sigma_{aK}) = 0 \tag{3.9}$$

or

$$\begin{aligned} [({X_\sigma}^2 + & R^2 N^2) \big((\frac{\partial}{\partial X_\mu})^2 + R^{-2} (\frac{\partial}{\partial N})^2 \big) \\ & - (X_\mu \frac{\partial}{\partial X_\mu} + N \frac{\partial}{\partial N})^2 - (D-1)(X_\mu \frac{\partial}{\partial X_\mu} + N \frac{\partial}{\partial N})] D(X, N) = 0, \end{aligned} \tag{3.9$'$}$$

with D being the dimension of space-time X_μ.

In this way, we have arrived at D_0 brane field equation with a nontrivial form beyond simple Klein-Gold equation, (3.9) or (3.9)', which reflects noncommutative space-time structure proper to YSTA. In fact, one finds that a simple Fourier analysis does not work well to find its solutions, on account of Lorentz-covariant couplings among individual degrees of freedom of space-time coming from their noncommutativity. Fourier transform of D_0 brane field turns out to satisfy entirely the same form of differential equation as (3.9), reflecting a distinctive invariance of the latter field equation under exchange, $\Sigma_{aK} \leftrightarrow \frac{\partial}{\partial \Sigma_{aK}}$.

Furthermore, one should notice that the field equation (3.9)' satisfies invariance under infinitesimal translations, $X_\mu \rightarrow X_\mu + \alpha_\mu N,\ N \rightarrow N - \alpha_\mu X_\mu / R^2$ corresponding to (2.19) and (2.20), and tends to a familiar massless Klein-Gordon equation in the contraction limit, (2.21) under the condition (2.22), that is, in the quantum-mechanical limit of YSTA stated at the end in section 2.

One can confirm by elementary calculations that the field equation admits, instead of familiar Fourier series with respect to individual coordinates Σ_{aK}, a series of Lorentz-invariant solutions with a continuous parameter α

$$\begin{aligned} &\exp\ i\alpha\ {\Sigma_{aK}}^2 \\ &= \exp\ i\alpha\ (X_\mu^2 + R^2 N^2)/\lambda^2, \end{aligned} \tag{3.10}$$

whose existence turns out to reflect scale invariance of the field equation (3.9) under $\Sigma_{aK} \rightarrow \rho\ \Sigma_{aK}$. Therefore, as a result of Fourier integral of the above series of solutions, one finds that any function of ${\Sigma_{aK}}^2$ can be a solution of the field equation.

There exists a special solution

$$\begin{aligned} (2\pi\lambda)^{-2} \int_{-\infty}^{+\infty} d\alpha\ \exp i\alpha\ (X_\mu^2 + R^2 N^2)/\lambda^2 &= \delta\ (\ X_\mu^2 + R^2 N^2) \\ &= \delta\ (X_\mu^2 + \lambda^2 \Sigma_{ab}^2). \end{aligned} \tag{3.11}$$

It reminds us of a regularized $D(x)$ function, $\tilde{D}(x)$, early proposed by Markov,[9]

$$\begin{aligned} \tilde{D}(x) &\equiv -\epsilon(x_0)\ /(2\pi)\ \delta({x_\mu}^2 + a^2) \\ &(= \int d\kappa^2\ [\ \delta(\kappa^2) - \frac{a}{2\kappa}\ J_1(a\kappa)\ ]\ \Delta(x;\kappa^2)), \end{aligned} \tag{3.12}$$

where one finds that light-cone singularity of conventional $\Delta(x;\kappa^2)$ function is shifted to time-like region by a certain universal length a corresponding to λ in the present approach.

4. Concluding Remarks

In this work we have studied a possible connection between noncommutative field theory on Yang's quantized space-time algebra (YSTA) and Matrix Model. We have found several important facts to support this possibility. First of all, we showed that the so-called kinetic term and potential term of D-particles of the matrix model are described covariantly and geometrically in terms of Casimir operator of $SO(D,1)$, a subalgebra of the primary algebra $SO(D+1,1)$ underlying YSTA, while they are usually understood to originate in Yang-Mills gauge theory.

In fact, it is well known that infinite-dimensional matrix representation of noncommutative position coordinates of D-particles in the matrix model has the origin in $U(N)$ Yang-Mills gauge symmetry caused by infinite N D-particles.[10] It should be emphasized, however, that M-theory as the matrix model of D-particles must be ultimately a master theory which rather underlies string theory and hence Yang-Mills gauge symmetry itself, because D-particles in the matrix model are regarded as fundamental constituents of strings.

In this connection, it is important to note that Hilbert space I defined in section 2, can be regarded as a representation space either of unitary infinite dimensional realization of YSTA or $U(N)$ gauge group with a large N limit in the matrix model.

In section 3, with aim of seeking for a short distance behavior of the present noncommutative field theory on YSTA, we have derived the field equation of D_0

brane on YSTA with a nontrivial form, (3.9) or (3.9)', by means of covariant Moyal star product (3.7) proper to YSTA, which clearly reflects the quantized noncommutative space-time structure of YSTA. Indeed, to solve the equation, a simple Fourier analysis does not work well on account of couplings among individual degrees of freedom of noncommutative space-time. Fourier transform of D_0 brane field tends to satisfy entirely the same form of differential equation as (3.9), reflecting a distinctive invariance of the latter field equation under exchanges, $\Sigma_{aK} \leftrightarrow \frac{\partial}{\partial \Sigma_{aK}}$. The situation may be well understood also from the fact that Fourier transform of a special solution exp $(i\alpha {\Sigma_{aK}}^2)$ given in (3.10), that is, $F_\alpha(k) \equiv \int \cdots \int \, (d\Sigma)^{D+1} \exp\,(ik_M \Sigma_{aM}) \exp\,(i\alpha {\Sigma_{aK}}^2)$ is also given in a form $\sim \exp -(i{k_L}^2/4\alpha)$.

Furthermore, we showed that the field equation (3.9)' has invariance under translations and scale transformation besides Lorentz-invariance and turns to a familiar massless Klein-Gordon equation in the quantum-mechanical limit of YSTA, (2.21) and (2.22), in which commutative space-time, Heisenberg uncertainty relation, translations and so forth in the ordinary quantum mechanics are all restored.

By preliminary considerations, we have found a series of Lorentz-covariant solutions with a continuous parameter α, (3.10), and a special invariant function (3.11), which resembles a modified Δ function, $\tilde{D}$, early proposed by Markov[9] towards divergence-free quantum field theory. The latter function which is free from light-cone singularity is given in terms of superposition of a series of $\Delta(x, \kappa^2)$ functions of fields with continuous mass κ and indefinite metric, as seen in (3.12).

According to the matrix model, Lagrangian of D_0 branes (3.1) and hence our field equation (3.9) must already include interactions among D_0 branes except those through super-partners which were entirely neglected in the present paper.*) Extensive studies on our field equation (3.9) and its supersymmetric extension have a vital importance, but must be left in future.

*) Supersymmetric YSTA can be easily realized by upgrading $(D+2)$-dimensional parameter space (q_M) satisfying (2.1) to the so-called superspace (q, θ).

Acknowledgements:

The author would like to thank participants in Meeting on "Mathematical Logic of Quantum System and its Application to Quantum Computer" for valuable discussions and suggestions, which was held at Research Institute for Mathematical Sciences, Kyoto University, in Feb. 2004.

REFERENCES

1. T. Banks, W. Fischler, S. H. Shenker and L. Susskind, "M Theory As A Matrix Model: A Conjecture," Phys. Rev. **D55** (1997) 5112, hep-th/9610043.

2. A. Connes, M.R. Douglas and A. Schwarz, "Noncommutative Geometry and Matrix Theory: Compactification on Tori," JHEP **02**, 003, hep-th/9711162; S. Minwalla, M.V. Raamsdonk and N. Seiberg, "Noncommutative Perturbative Dynamics," JHEP **02**,020, hep-th/9912072; M. Chaichian, A. Demichev and P. Prešnajder, "Quantum Field Theory on Noncommutative Space-Times and the Persistence of Ultraviolet Divergences," Nucl. Phys. **567** (2000) 360, hep-th/9812180; R.J. Szabo, "Quantum Field Theory on Noncommutative Spaces," Phys.Rev. **378** (2003) 207, hep-th/0109162; M.R. Douglas and N.A. Nekrasov, "Noncommutative Field Theory," Rev.Mod.Phys. **73** (2001) 977, hep-th/0106048; C.E. Carlson and C.D.Carone, "Noncommutative Gauge Theory without Lorentz Violation," Phys. Rev. **D66** (2002)075001, hep-th/0206035; H. Kase, K. Morita, Y. Okumura and E. Umezawa, "Lorentz-invariant Non-Commutative Space-Time Based On DFR Algebra," hep-th/0212176.

3. S. Tanaka, "Space-time quantization and matrix model," Nuovo Cim.**114 B** (1999) 49, hep-th/9808064.

4. S. Tanaka, "Space-Time Quantization and Nonlocal Field Theory - Relativistic Second Quantization of Matrix Model-," hep-th/0002001.

5. S. Tanaka, "Yang's Quantized Space-time Algebra and Holographic Hypothesis," hep-th/0303105.

6. C.N. Yang, Phys.Rev.**72** (1947) 874; Proc. of International Conf. on Elementary Particles, 1965 Kyoto, pp 322-323.
7. H.S. Snyder, Phys.Rev.**71** (1947) 38; **72** (1947), 68.
8. N.Ja. Vilenkin and A.U. Klimyk, "Representation of Lie groups and special functions II," Kluwer Academic Publishers, 1993.
9. M.A. Markov, Nucl. Phy. **10** (1959) 140.
10. E. Witten, "Bound States of Strings and p-Branes," Nucl. Phys. **B460** (1995) 335, hep-th/9510135.